NEW TYPES OF CONTINUOUS FUNCTIONS IN IDEAL TOPOLOGICAL SPACES

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ABSTRACT

An ideal I is a nonempty collection of subsets of a topological space (X,τ) closed under heredity and finite additivity. The main objective of this paper is to introduce new types of continuous functions in ideal topological spaces (X,τ,I), and obtain several characterizations and some properties of these functions. Also, we investigate its relationship with other types of functions.

KEYWORDS: I-open set, α-I-open set, α-I-continuous function, semi-I-continuous function.

1. INTRODUCTION

The notion of α-open sets was first introduced and investigated by Njastad [1]. By using α-open sets, Mashhour et al. [2] defined and studied α-continuity and α -openness in topological spaces. Quite recently, Hatir and Noiri [3] have introduced the notion of α-I-continuous functions and used it to obtain a decomposition of continuity. In this paper, we obtain several characterizations and study of α-I-continuous functions. In 2000 Navalagi [4] Introduced the concept of semi-α-open sets in topological spaces, Through this concept we introduce new concept called semi-α-I-open sets in ideal topological spaces. Moreover, we introduce the concept semi-α-I-continuous functions. Throughout this paper Cl (A) and Int (A) denote the closure and the interior of A, respectively. Let (X,τ) be a topological space and I an ideal of subsets of X. An ideal is defined as a nonempty collection I of subsets of X satisfying the following two conditions: (1) If $A \in I$ and $B \subset A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$. An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). For a subset $A \subset X$, $A^*(I) = \{x \in X \mid U \cap A \notin I$ for each neighbourhood U of x$\}$ is called the local function of A with respect to I and τ [5]. We simply write A^* instead of $A^*(I)$ in case there is no chance for confusion. X^* is often a proper subset of X. The hypothesis $X = X^*$ [6] is equivalent to the hypothesis $\tau \cap I = \varnothing$ [7]. For every ideal topological space (X, τ, I), there exists a topology $\tau^*(I)$, finer than τ, generated by $\beta(I,\tau) = \{U \backslash I : U \in \tau$ and $I \in I\}$, but in general $\beta(I,\tau)$ is not always a topology [8]. Additionally, $Cl^*(A) = A \cap A^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

2. PRELIMINARIES

First we recall some definitions used in the sequel.

Definition 2.1:
A subset A of an topological space (X, τ) is said to be:

I. **α-open set [1]if A $\subseteq$ int(cl(int(A))).**

II. **semi-α-open set [4] if U$\subseteq$ A $\subseteq$cl(U) and U is α-open in (X,τ).**

III. **semi-open set [9] if A$\subseteq$ cl(int(A))).**

Complement of the above mentioned open sets are called their respective closed sets.

Definition 2.2:
A subset A of an ideal topological space (X, τ , I) is said
to be:
(1) α-I-open set **[3]** if A $\subseteq$ Int (Cl*(Int (A)))

(2) semi-I-open set **[3]** if A $\subseteq$ Cl* (Int (A))

(3) I-open set **[10]** if A $\subseteq$ Int(A*)

(4) semi-α-I-open set if U $\subseteq$A $\subseteq$cl*(U)and U is α-I-open set.

The family of all α-I-open (resp. semi-I-open, I-open, semi- α-I-open) is denoted by αIO (X),(resp. SIO(X),IO(X),SαIO(X).

Proposition 2.1

i) Every α-I-open sets is α-open sets.

(ii) Every semi-I-open set is semi-open set.

Proof:

(i) Let A be an α-I-open set. Then we have

A $\subseteq$ Int (Cl*(Int (A))) = Int((Int(A))*$\cup$Int(A))

$\subseteq$ Int(Cl(Int(A)) $\cup$Int(A)) $\subseteq$ Int(Cl(Int(A))).

This show that A is α-open set .

(ii) Let A be a semi-I-open set. Then we have A $\subseteq$ Cl* (Int (A)) $\subseteq$ (Int(A))* $\cup$ Int (A)$\subseteq$ Cl(Int(A)) $\cup$ Int(A) = Cl(Int(A)) . This show that A is semi-open set .

Proposition 2.2: Every open set of an ideal topological space is α-I-open set.

Proof:

Let A be any open set. Then we have

$A=Int(A)\subseteq Int((Int(A))^*\cup Int(A))=Int(Cl^*(Int(A)))$.

This show that A is α-I-open set .

The following example shows that the converse of the above proposition is not true.

Example 2.1:

Consider the ideal topological space (X, τ ,I),where X={1,2,3,4}

with $\tau=\{\varnothing,X,\{1\},\{2\},\{1,2\},\{1,2,3\}\}$ and I={ $\varnothing$,{2}}.In this ideal space the set A={1,2,4} is α-I-open set but not open set because of $A\notin\tau$.

Proposition 2.3: Every α-I-open set of an ideal topological space is semi-α-I-open set.

Proof: Let A be any α-I-open set.

Since $A\subseteq A$ and $A\subseteq Cl^*(A)\Rightarrow A\subseteq A\subseteq Cl^*(A)$

This show that A is semi-α-I-open set.

The following example shows that the converse of the above proposition is not true.

Example 2.2:

Consider the ideal topological space (X, τ , I),where X={a,b,c,d}

with $\tau=\{\varnothing,X,\{a\},\{b\},\{a,b\},\{a,b,c\},\{a,b,d\}\}$ and I={ $\varnothing$,{c}}.In this ideal space the set A={b,c,d} is semi-α-I-open set but not α-I-open set

because :

$A=\{b,c,d\}\ \Rightarrow\ int\,\{b,c,d\}=\{b\},\ \{b\}^*=\{b,c,d\}$

$cl^*\{b\}=\{b\}\cup\{b\}^*=\{b\}\cup\{b,c,d\}=\{b,c,d\}$

$cl^*\{b\}=\{b,c,d\}\ \Rightarrow\ int\,(cl^*\{b,c,d\})=\{b\}$

$\Rightarrow\ A\not\subset int(cl^*(int(A)))=\{b\}$

$\Rightarrow$ A is not α-I-open set.

Remark 2.1:

It is clear that, I-open set and open set are independent concepts.

Example 2.3:

Let X={a,b,c,d} with a topology $\tau=\{\varnothing,X,\{c\},\{a,b\},\{a,b,c\}\}$

and I={$\varnothing$,{a}}.Then $\{b,c,d\}\in IO(X,\tau)$ but $\{b,c,d\}\notin\tau$.

Example 2.4:

Let X be as in Example 2.3, $\tau=\{\varnothing,X,\{d\},\{a,c\},\{a,c,d\}\}$

and I={$\varnothing$,{c},{d},{c,d}}.It is clear that $\{a,c,d\}\in\tau$,but $\{b,c,d\}\notin IO(X,\tau)$.

Remark 2.2:

The intersection of two I-open sets need not be I-open as is illustrated by the following example.

Example 2.5:

Let X={a,b,c,d}, $\tau=\{\varnothing,X,\{a,b\},\{a,b,c\}\}$ and I={$\varnothing$}.

Then {a,c},{b,c,d} $\in$ IO(X,τ), but {a,c} $\cap$ {b,c,d} $\notin$ IO(X,τ).

Proposition 2.4: Let (X,τ,I) be an ideal topological space. If I={$\varnothing$} (resp. P(X)),then αIO(X)=τ_α (resp. τ = αIO(X))

Proof:

(1) Let I = {$\varnothing$}. Then A* = Cl (A) and hence Cl*(A) = A $\cup$ A* =Cl(A) for every subset A of X. Therefore, Int (Cl*(Int(A)))=Int(Cl(Int(A))) and hence αIO(X) = τ_α.

(2) Let I = P(X), then A* =$\varnothing$, and Cl*(A) = A for every subset A of X. Therefore, Int (Cl*(Int(A)))=Int(Int(A))=Int(A) and hence τ = αIO(X).

3. I-continuous function, α-I-continuous function and semi-α-I-continuous function

Definition 3.1[10]: A function f **:** (X,τ, I) → (Y, σ) is said to be I-continuous if for every V$\in\sigma$, f^{-1}(V) $\in$IO (X,τ).

The following two examples show that the concept of continuity and I-continuity are independent .

Example 3.1:

Let X = Y = {a, b, c} , τ = {X,$\varnothing$,{a},{c},{a,b},{a,c}},

I = {$\varnothing$, {b }, {c }, {b, c}} on X and σ = {Y,$\varnothing$,{a},{c},{a,c}}. Then the identity function f : (X,τ, I) → (Y,σ) is continuous but not I-continuous because {c} $\in\sigma$, but f^{-1}({c}) = {c} $\notin$ IO(X).

Example 3 .2:

Let X = Y = {a,b,c}, τ = σ = {X,$\varnothing$,{a}} and

I ={$\varnothing$,{b}} on X. Then f : (X,τ,I) →(Y,σ) which is defined by:

f (a) = a = f (b) and f (c) = c is I-continuous but not continuous.

Definition 3.2 [3]: A function f **:** $(X, \tau, I) \to (Y, \sigma)$ is called α-I-continuous if the inverse image of each open set of Y is α-I-open .

Definition 3.3: Let N be a subset of a space (X,τ, I) and let x ∈X. Then N is called α-I-neighborhood of x, if there exists a α-I-open set U containing x such that U $\subseteq$N.

Definition 3.4 : A function f **:** $(X, \tau, I) \to (Y, \sigma)$ is called semi-α-I-continuous if the inverse image of each open set of Y is semi-α-I-open .

Theorem 3.1: For a function $f : (X, \tau, I) \to (Y, \sigma)$ the following are equivalent:

(i) f is α-I-continuous function.

(ii) For each x $\in$ and each V $\in \sigma$ containing f (x), there exists W$\in$αIO(X) containing x such that f (W) $\subset$V.

(iii) For each x $\in$ and each V $\in \sigma$ containing f (x) , (f^{-1}(V))* is a α-I-neighborhood of x.

Proof: (i) $\Rightarrow$ (ii) : Since V $\in \sigma$ containing f (x), then by (i), f^{-1} (V) $\in$ αIO(X) , by putting W = f^{-1}(V) which containing x. Therefore f (W) $\subset$ v

(ii) $\Rightarrow$ (iii): Since V $\in \sigma$ containing f (x), then by (ii), there exists W $\in$ αIO(X) containing x such that f(W) $\subset$V. So, x$\in$ W $\subseteq$ int(W*) $\subseteq$ int(f^{-1}(V))* $\subseteq$(f^{-1}(V))* . Hence (f^{-1} (V))* is a neighborhood of x

(iii) $\Rightarrow$ (i) : Obvious

Remark 3.1:

Every continuous function is α-I-continuous function, so it is semi-α-I-continuous function. But the converse is not true in general.

Example 3.3:

Let X ={a,b,c,d}, $\tau = \{\varnothing, X, \{a\}\}$, I={$\varnothing$,{c}},Y={x,y,z}, $\sigma = \{\varnothing, Y, \{x\}\}$,Then $f : (X, \tau, I) \to (Y, \sigma)$ which is defined by:

f (a) = f (b)= x , f (c) = y and f (d) = z

is α-I-continuous but not continuous, because $\{x\} \in \sigma$, but $f^{-1}(\{x\}) = \{a,b\} \notin \tau$.

Also, f is semi-α-I-continuous function but not continuous, because $\{x\} \in \sigma$, but $f^{-1}(\{x\})$ $=\{a,b\} \notin \tau$.

Remark 3.2:

Every α-I-continuous function is semi-α-I-continuous function. But the converse is not true in general.

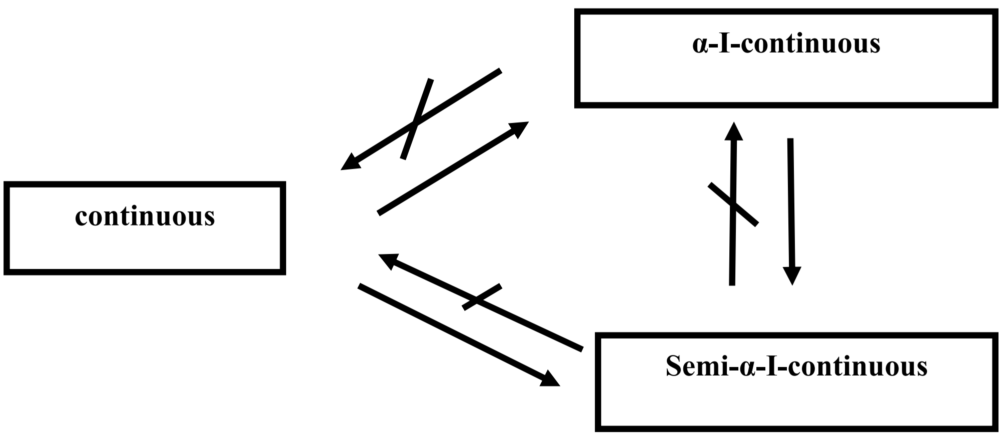
Example 3 .4:

Let X ={1,2,3 }, τ = {∅, X,{1},{2},{1,2}}, I={∅,{3}}

Then f : X → X which is defined by:

f (1) = 1 , f (2) = f (3) = 2 is semi-α-I-continuous but not α-I-continuous. because $\{2\} \in \tau$, but $f^{-1}(\{2\}) = \{2,3\} \notin \alpha IO(X)$.

Through the above remarks we get to the diagram following



Remark3.3:

Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two functions, then:

If f and g are α-I-continuous, then $g \circ f : X \longrightarrow Z$ need not to be α-I-continuous as the following example shows:

Example 3.5 :

Let X = {1, 2, 3, 4}, τ_X= {∅, X,{3},{1, 3},{1, 2, 3}}, I={∅,{2}}

Let Y = {a, b, c}, τ_Y= {∅, Y ,{c}},

Define $f : X \longrightarrow Y$ by $f(1) = f(2) = a$, $f(3) = f(4) = b$

Define $g : Y \longrightarrow X$ by $g(a) = g(c) = 3$, $g(b) = 1$

It is easily seen that f and g are α-I-continuous, but $g \circ f$: X ⟶X, where $g \circ f$ (1) $= g \circ f$ (2) = 3, $g \circ f$ (3) $= g \circ f$ (4) = 1, hence $g \circ f$ is not α-I-continuous since {3} is open set in X, but $(g \circ f)^{-1}\{3\} = \{1, 2\}$ is not α-I-open set in X.

Proposition 3.1: Let f : (X, τ ,I) →(Y, σ ,J) be α-I-continuous function and g : (Y, σ ,J) →(Z, μ) be continuous. Then $g \circ f$ is α-I-continuous function.

Proof:

Let W ∈μ . Then $(g \circ f)^{-1}(W) = (f^{-1} \circ g^{-1})(W) = f^{-1}(g^{-1}(W))$ .Since $g^{-1}(W)$ is open as g is continuous. Now since f is α-I-continuous function. So $f^{-1}(g^{-1}(W))$ is α-I-open set. Hence $g \circ f$ is α-I-continuous function.

4. α*-I-continuous function and semi-α*-I-continuous function

Definition 4.1: A function f : (X, τ , I) → (Y, σ) is called α*-continuous if the inverse image of each α-open set of Y is α-open .

Remark4.1:

Every α*-continuous function is α-continuous function but the converse is not true in general as the following examples show.

Example 4.1:

Let X = {a, b, c, d}, τ_X= {X, ∅, {a}, {b}, {a, b}, {a, b, c}} $\tau_{\alpha(X)} = \tau_{(X)} \cup$ {{a, b, d}}

Define f : X ⟶X by f (a) = a, f (b) = f (c) =c, f (d) = d.

It is easily seen that f is α-continuous but it is not α*-continuous since {a, b, d} $\in \tau_{\alpha(X)}$ but $f^{-1}(\{a, b, d\}) = \{a, d\} \notin \tau_{\alpha(X)}$

Hence f is α-continuous but it is not α*-continuous.

Definition 4.2: A function f : (X, τ , I) → (Y, σ) is called α*- I-continuous if the inverse image of each α-I-open set of Y is α-I-open .

Definition 4.3: A function f : (X, τ , I) → (Y, σ) is called semi-α*- I-continuous if the inverse image of each semi-α-I-open set of Y is semi-α-I-open .

Remark 4.2:

The concepts of α*-I-continuity and semi-α*-I-continuity are independent as the following examples show:

Example 4.2:

Let X = {a, b, c, d}, τ_X= {X, ∅, {a}, {b}, {a, b}, {a, b, c}},I={∅,{c}},

Let Y = {a, b, c}, τ_Y= {Y, ∅, {a}, {b}, {a, b}},

Define $f : X \longrightarrow Y$ by $f(a) = f(d) = b$, $f(c) = c$, $f(b) = a$

It is easily seen that is semi-α*-I-continuous but it is not α*-I-continuous, since {b} is semi-α-I-open set in Y , but $f^{-1}(\{b\}) = \{a, d\}$ is not semi-α-I-open set in X.

Example 4.3:

Let X = {a, b, c, d}, τ_X= {X, ∅, {a},{b}, {a, b},{a, b, c}}, I={∅,{c}},

Define $f : X \longrightarrow X$ by $f(a) = f(b) = b$, $f(c) = d$ and $f(d) = c$

It is easily seen that f is α*-I-continuous function but it is not Semi-α*-I-continuous function ,{a, c} is semi-α-I-open set but $f^{-1}(\{a, c\}) = \{d\}$ is not semi-α-I-open set in X.

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